J. Fluid Mech. (1969), *vol.* 39, *part* 2, *pp.* 257–267
Printed in Great Britain

The model of a single infinite disk oscillating about a state of steady rotation

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(Received 14 May 1969)

The flow produced by a disk performing small oscillations in a rotating system is considered. Results are obtained for the first-order harmonic velocity and the second-order steady velocity. It is then shown that this mathematical model does not always represent an axially bounded fluid in the limit of infinite separation, and to be general one must allow a steady, azimuthal, perturbation velocity with an arbitrary value to exist at infinity.

1. Introduction

Hunt & Johns (1963) discussed the boundary-layer flow produced over a smooth sea bed by tidal or gravity waves. They began by neglecting effects due to the earth's curvature, assuming that the earth could be considered as flat about any given latitude while retaining any effects due to the earth's rotation. They also restricted their waves to be harmonic in time with a small amplitude of oscillation which allowed a first approximation to be obtained by linearizing the equations of motion. They thus showed that, to highest order, the balance of forces in the boundary layer was between the viscous, accelerative and Coriolis forces. Their solution, apart from its possible practical application, was interesting because of a singularity that it contained. This occurred when the latitude took a certain, specific value (the actual value depended on the strength of the earth's rotation and on the frequency of the wave), whereupon a solution could not be obtained by linearization techniques alone. Although they noted this effect, Hunt & Johns did not pursue their analysis of this singularity any further.

Benney (1965) examined another problem, which is in many respects directly analogous to the one just described. We shall now describe this problem in some detail, since it will be considered further in this paper. A semi-infinite fluid of density ρ and kinematic viscosity ν lies above an infinite disk. Both the disk and the fluid are assumed to be in a state of uniform rotation (angular velocity Ω) about an axis normal to the disk. Small torsional oscillations of magnitude ω, frequency σ, are then superimposed on the basic rotation of the disk. The oscillations are small in the sense that $\omega \ll \Omega$, which, as we shall see, implies that the Coriolis force will always be of more importance than the centrifugal force. The problem is to determine the fluid motion when a purely periodic state has been reached.

As is immediately apparent from the condition of no-slip, the effect of oscillating the solid boundary is to produce near the boundary additional azimuthal and radial velocity components with the same frequency and order of magnitude as the disk's oscillations. To satisfy continuity requirements, it is then necessary for a small axial velocity to come into existence and to persist into the far field. These harmonic terms are only first approximations to the exact solution and they interact in the non-linear terms of the equations of motion to produce velocity components which are steady or a multiple of the basic perturbation velocity. Of particular interest are the steady components of the velocity. This is partly because the mean flow often has a greater significance than the oscillatory motion and partly because the equation for the steady motion is singularly different in nature to the equations for the time-harmonic components of the flow.

To demonstrate the difference, the special case of $\sigma \to 0$ when the problem approaches a steady state is discussed in detail. From the results obtained it is deduced that the mathematical model may not always correspond to the intended physical situation. That is, the state of basic rotation Ω, which has been proposed for the disk and fluid, need not necessarily be the limit of a problem in which a boundary in the far field also rotates with angular velocity Ω. To investigate this further, a second disk, parallel to the first one and with the same rotation, is introduced to bound the fluid a finite distance away. The results show that if the single-disk problem is to correspond to the two-disk problem with large separation, the boundary conditions for the single-disk problem must be amended to allow for the possibility of a steady, azimuthal velocity in the far field.

2. The equations of motion

Cylindrical co-ordinates $(\tilde{r}, \tilde{\theta}, \tilde{z})$ are chosen with accompanying velocity components $(\tilde{u}, \tilde{v}, \tilde{w})$. The disk is defined by the equation $\tilde{z} = 0$ with $\tilde{r} = 0$ as its axis of rotation. Assuming axial symmetry, the equations of motion are:

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{r}} + \tilde{w}\frac{\partial \tilde{u}}{\partial \tilde{z}} - \frac{\tilde{v}^2}{\tilde{r}} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{r}} + \nu\left[\nabla^2 \tilde{u} - \frac{\tilde{u}}{\tilde{r}^2}\right], \tag{2.1}$$

$$\frac{\partial \tilde{v}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{r}} + \tilde{w}\frac{\partial \tilde{v}}{\partial \tilde{z}} + \frac{\tilde{u}\tilde{v}}{\tilde{r}} = \nu\left[\nabla^2 \tilde{v} - \frac{\tilde{v}}{\tilde{r}^2}\right], \tag{2.2}$$

$$\frac{\partial \tilde{w}}{\partial \tilde{t}} + \tilde{u}\frac{\partial \tilde{w}}{\partial \tilde{r}} + \tilde{w}\frac{\partial \tilde{w}}{\partial \tilde{z}} = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial \tilde{z}} + \nu\nabla^2 \tilde{w}, \tag{2.3}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{\tilde{u}}{\tilde{r}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0, \tag{2.4}$$

where $\tilde{t}$ is time and $\tilde{P}$ pressure. The boundary conditions are:

$$\left.\begin{aligned} \tilde{u} &= \tilde{w} = 0, \\ \tilde{v} &= \Omega\tilde{r} + 2\omega\tilde{r}\,e^{i\sigma\tilde{t}}, \end{aligned}\right\} \quad \text{on} \quad \tilde{z} = 0, \quad \text{all} \quad \tilde{r}, \tilde{t}, \tag{2.5a}$$

and

$$\left.\begin{aligned} \tilde{u} &\to 0, \\ \tilde{v} &\to \Omega\tilde{r}, \end{aligned}\right\} \quad \text{as} \quad \tilde{z} \to \infty, \quad \text{all} \quad \tilde{r}, \tilde{t}. \tag{2.5b}$$

In using complex notation it is to be understood that the real part of the equation is to be taken. The factor of two in the second boundary condition is a matter of convenience.

Three simplifications of these equations are now possible. The first is an assumption, drawn from the geometry of the problem, that the radial variable $\tilde{r}$ can be eliminated as a similarity variable. Secondly, the equation of continuity allows the introduction of a Stokes stream function. Thirdly, it is more convenient to change the co-ordinate axes to a rotating system with angular velocity Ω. These considerations motivate the following transformations:

$$\left.\begin{aligned} \tilde{u} &= \tilde{r}\frac{\partial \tilde{F}}{\partial \tilde{z}}(\tilde{z},\tilde{t}), & \tilde{w} &= -2\tilde{F}(\tilde{z},\tilde{t}), \\ \tilde{v} &= \Omega\tilde{r} + 2\tilde{r}G(\tilde{z},\tilde{t}), & \frac{\tilde{P}}{\rho} &= \tfrac{1}{2}\tilde{r}^2\tilde{P}_1(\tilde{t}) + \tilde{P}_2(\tilde{z},\tilde{t}). \end{aligned}\right\} \tag{2.6}$$

Equations (2.1) and (2.2) become

$$\frac{\partial^2 \tilde{F}}{\partial \tilde{z}\,\partial \tilde{t}} + \frac{\partial \tilde{F}}{\partial \tilde{z}}\frac{\partial \tilde{F}}{\partial \tilde{z}} - 2\tilde{F}\frac{\partial^2 \tilde{F}}{\partial \tilde{z}^2} - 4\Omega\tilde{G} - 4\tilde{G}^2 = \Omega^2 - \tilde{P}_1 + \nu\frac{\partial^3 \tilde{F}}{\partial \tilde{z}^3}, \tag{2.7}$$

$$\frac{\partial \tilde{G}}{\partial \tilde{t}} + 2\left(\tilde{G}\frac{\partial \tilde{F}}{\partial \tilde{z}} - \tilde{F}\frac{\partial \tilde{G}}{\partial \tilde{z}}\right) + \Omega\frac{\partial \tilde{F}}{\partial \tilde{z}} = \nu\frac{\partial^2 \tilde{G}}{\partial \tilde{z}^2}. \tag{2.8}$$

Equation (2.3) serves only to determine the vertical pressure gradient $\tilde{P}_2$, while (2.4) is automatically satisfied. Furthermore, to satisfy the boundary conditions at infinity, (2·5*b*), it follows that

$$\tilde{P}_1(\tilde{t}) = \Omega^2. \tag{2.9}$$

Finally, we introduce dimensionless variables. With an infinite fluid, any length scale has to be based on the viscosity ν, and on physical considerations one would expect two particular length scales to be important. These are the Stokes layer thickness, $(\nu/\sigma)^{\frac{1}{2}}$, and the Ekman layer thickness $(\nu/\Omega)^{\frac{1}{2}}$. The former would be more important if $\sigma \gg \Omega$, when the viscous force would balance the acceleration; the latter would be important if $\Omega \gg \sigma$, when the viscous force would balance the Coriolis force. In fact, it turns out that the lengths which arise are combinations of these $(\nu/2\Omega \pm \sigma)$, and as a result there is no basic difference in employing either. We shall actually choose the Ekman layer thickness as the natural length scale for the system. Hence,

$$\left.\begin{aligned} &\tilde{z} = \left(\frac{\nu}{2\Omega}\right)^{\frac{1}{2}} z, \quad \tilde{t} = \sigma^{-1}t, \quad \tilde{G} = \omega G, \\ &\tilde{F} = \omega\left(\frac{2\nu}{\Omega}\right)^{\frac{1}{2}} F, \quad \frac{\partial \tilde{F}}{\partial \tilde{z}} = 2\omega\frac{\partial F}{\partial z}. \end{aligned}\right\} \tag{2.10}$$

Under these transformations, (2.7) and (2.8) become

$$F''' - p\frac{\partial F'}{\partial t} + G = \epsilon(F'^2 - 2FF'' - G^2), \tag{2.11}$$

$$G'' - p\frac{\partial G}{\partial t} - F' = 2\epsilon(GF' - FG'), \tag{2.12}$$

where ϵ and p are dimensionless parameters defined by

$$\epsilon = \frac{\omega}{\Omega}, \quad \text{and} \quad p = \frac{\sigma}{2\Omega}, \tag{2.13}$$

and a dash represents a differentiation with respect to z. These equations are to be solved subject to the boundary conditions,

$$\left.\begin{aligned} F = \partial F/\partial z = 0, \quad G = e^{it} \quad \text{at} \quad z = 0, \quad \text{all } t. \\ \partial F/\partial z \to 0, \quad G \to 0 \quad \text{as} \quad z \to \infty, \quad \text{all } t. \end{aligned}\right\} \tag{2.14}$$

3. The perturbation solution

We now look for an asymptotic solution as $\epsilon \to 0$. To do this, one would normally assume formal power series expansions in ϵ for F and G, and proceed from there. It is more convenient, however, to note that because of the boundary conditions, the first-order solution would be simple harmonic in time, the second would be steady and second harmonic, the third simple and third harmonic, and so on. Consequently, a more useful expansion is

$$F \sim F_{01}(z)\,e^{it} + \epsilon[F_{10}(z) + F_{12}(z)\,e^{2it}] + \epsilon^2[F_{21}(z)\,e^{it} + F_{23}(z)\,e^{3it}] + \dots, \tag{3.1}$$

for F, with a similar expansion for G.

We will content ourselves with the calculation of the first-order solution and the second-order steady solution. To obtain the equations for F_{01} and G_{01}, (3.1) are substituted into (2.11) and (2.12) and terms are equated whose coefficient is ϵ to the power zero. Thus,

$$\left.\begin{aligned} F'''_{01} - piF'_{01} + G_{01} = 0, \\ G''_{01} - piG_{01} - F'_{01} = 0, \end{aligned}\right\} \tag{3.2}$$

while the boundary conditions are

$$F_{01}(0) = F'_{01}(0) = 0, \quad G_{01}(0) = 1, \quad F'_{01}(\infty) = G_{01}(\infty) = 0. \tag{3.3}$$

The solution is:

$$\left.\begin{aligned} F_{01} &= \frac{i}{2}\left[\frac{1}{\alpha}e^{\alpha z} - \frac{1}{\beta}e^{\beta z} + \left(\frac{1}{\beta} - \frac{1}{\alpha}\right)\right], \\ G_{01} &= \tfrac{1}{2}[e^{\alpha z} + e^{\beta z}], \end{aligned}\right\} \tag{3.4}$$

where α and β are defined by

$$\left.\begin{aligned} \alpha = [i(p+1)]^{\frac{1}{2}}, \\ \beta = [i(p-1)]^{\frac{1}{2}}, \end{aligned}\right\} \quad \text{negative real parts.}$$

The solution fails when $p = 1$. Physically, the cause of this can be traced to a resonance on feedback situation far from the disk (where viscosity is relatively unimportant) between the accelerative and Coriolis forces. The singularity corresponds exactly to the one discovered by Hunt & Johns. If a solution exists in this case, it can only be because the non-linear terms of the equations of motion eventually bound the far-field growth of the solution. Thus, any simple asymptotic expansion such as (3.1), which initially neglects the non-linear terms, will be unable to give a solution. Similar resonance situations occur for the harmonic $\epsilon$when $p^{nit} = 1/n$ (n = an integer). The case $p = 0$ is especially interesting, since it is near to an arbitrarily large number of such singularities. These will each be associated, however, with ϵ raised to the power n, and it is to be hoped that, for small ϵ, their effect for p close to zero can be neglected. We shall be assuming that $p \to 0$ is a regular limit.

The equations for F_{10} and G_{10} come from (2.11) and (2.12) after substitution of (3.1). The steady terms, whose coefficient is ϵ, are:

$$\left.\begin{aligned} F'''_{10} + G_{10} &= \tfrac{1}{2}F'_{01}F'_{01} - F_{01}F''_{01} - \tfrac{1}{2}G_{01}G_{01}, \\ G''_{10} - F'_{10} &= (G_{01}F'_{01} - F_{01}G'_{01}), \end{aligned}\right\} \tag{3.5}$$

with boundary conditions,

$$\left.\begin{aligned} F_{10}(0) = F'_{10}(0) = G_{10}(0) &= 0, \\ F'_{10}(\infty) = G_{10}(\infty) &= 0. \end{aligned}\right\} \tag{3.6}$$

The solution of these equations, although straightforward, is very complicated algebraically. It takes the form,

$$\left.\begin{aligned} F_{10} &= C + \text{terms that decay exponentially with } z, \\ G_{10} &= \text{terms that decay exponentially with } z, \end{aligned}\right\} \tag{3.7}$$

where C is a constant for any given value of p. (For further details, here and later, see Jones 1968.)

The value of C, regarded as a function of p, is interesting, since it represents the steady, axial velocity far from the disk. The numerical values have been calculated using a computer and are represented in figures 1 and 2 for the range $0 \leqslant p \leqslant 100$. Values of $C(p)$ that are positive represent an inflow towards the disk from infinity, and it can be seen that there is a transition point between inflow and outflow for $p \approx 4.3$. $C(p)$ has its maximum value at $p \approx 9$, and the change in nature of $C(p)$ for $p < 9$ and $p > 9$ is marked. The negative values of $C(p)$ are greater in magnitude than the positive values by a factor of about 10, and there is a gradient discontinuity at $p = 1$, the resonance point.

Benney in his paper gave only three limiting values, namely,

$$\left.\begin{aligned} C(p) &\to -\frac{3\sqrt{2}}{20}, \quad p \to 0, \\ C(p) &\to -\frac{29\sqrt{2}-30}{34}, \quad p \to 1, \\ C(p) &\to \frac{\sqrt{2}}{8p}, \quad p \to \infty, \end{aligned}\right\} \tag{3.8}$$

which agree with our results. However, it should be mentioned that the limit $p \to 1$ does not commute with the limit $z \to \infty$. Thus, if p is allowed to approach unity in the complete expression for F_{10}, including exponential terms, the value for the constant at infinity is $(137 - 116\sqrt{2})/34$. This is important if a proper treatment of the resonance phenomenon is attempted. We shall see later that the limit $p \to 0$ also has to be treated with care, because we are considering a mathematical model with a non-uniformity near to this value.

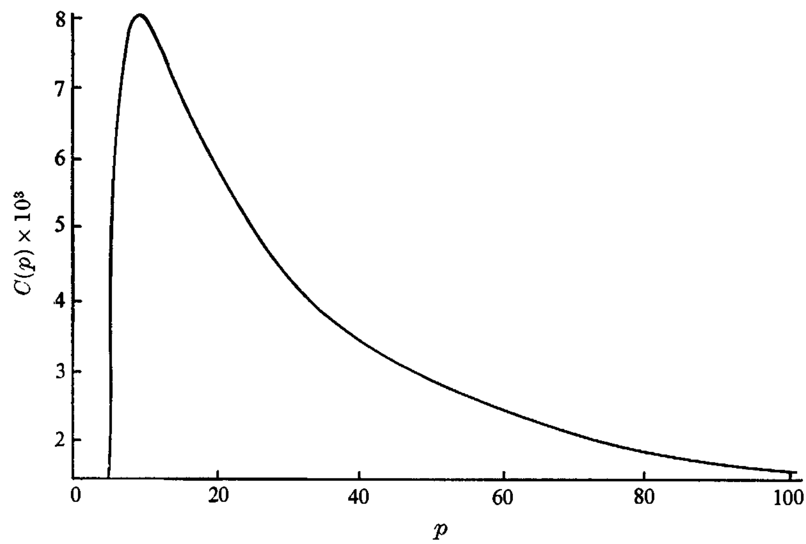

FIGURE 1

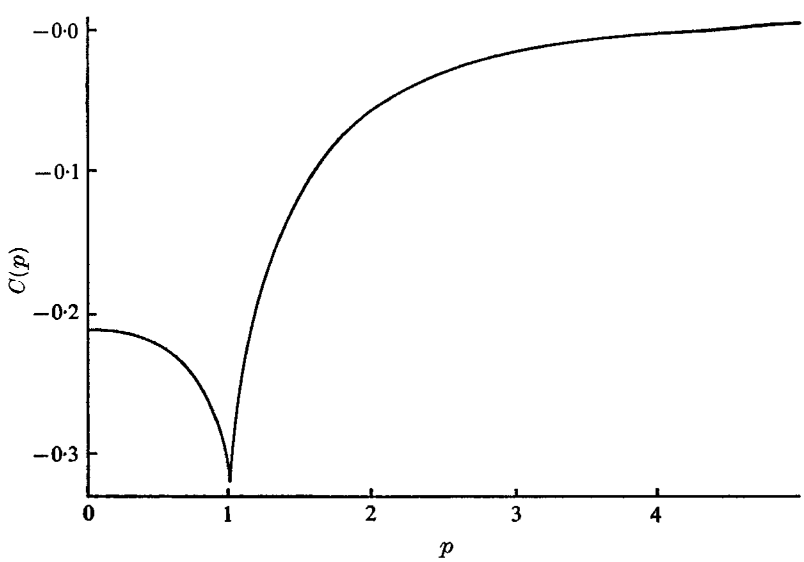

FIGURE 2

4. The two disk problem

When $p = 0$ the solution (3.4) becomes

$$\left.\begin{aligned} F_{01} &= -\frac{1+i}{2\sqrt{2}}\exp\left(-\frac{1+i}{\sqrt{2}}z\right)-\frac{1-i}{2\sqrt{2}}\exp\left(-\frac{1-i}{\sqrt{2}}z\right)+\frac{1}{\sqrt{2}},\\ G_{01} &= \frac{1}{2}\left\{\exp\left(-\frac{1+i}{\sqrt{2}}z\right)+\exp\left(-\frac{1-i}{\sqrt{2}}z\right)\right\}.\end{aligned}\right\} \tag{4.1}$$

Now $p = 0$ implies $\sigma = 0$, which means that the problem is time independent. The harmonic factors in the expansions (3.1) all reduce to unity when transformed to real co-ordinates. Thus, the solution (4.1) is the linearized flow solution that occurs when a disk rotates with a small, steady perturbation velocity above the basic state of rotation. That is, we have an Ekman layer. However, as is well known for this problem, if the fluid is bounded in the axial direction by some rigid surface, this boundary will not have the rotational speed of the interior of the fluid. In fact, the interior adopts some intermediate angular velocity between those of the two boundaries, and an Ekman layer forms at both surfaces. While this is easy to understand for a steady flow ($p = 0$), it is not clear how the fluid interior and some distant boundary will be connected in the general, oscillating disk problem ($p \neq 0$).

To investigate this, suppose there is a second disk in the fluid at $\tilde{z} = d$, which rotates with the basic angular velocity Ω. The equations of motion can be reduced to (2.7) and (2.8) as before, but we shall now introduce dimensionless variables using the length scale d:

$$\left.\begin{gathered}\tilde{z} = \frac{d}{\sqrt{2}}\zeta, \quad \tilde{t} = \sigma^{-1}t, \quad \tilde{G} = \omega\mathscr{G}, \quad \tilde{F} = \omega\sqrt{2}d\mathscr{F}, \\ \frac{\partial \tilde{F}}{\partial \tilde{z}} = 2\omega\frac{\partial \mathscr{F}}{\partial \zeta}.\end{gathered}\right\} \tag{4.2}$$

The radial pressure gradient cannot be eliminated as before but to simplify it, we can write

$$\tilde{P}_1(z,t) = \Omega^2 + 4\omega\Omega K(t). \tag{4.3}$$

Equations (2.7) and (2.8) then reduce to

$$\left.\begin{aligned} E\mathscr{F}''' - p\frac{\partial \mathscr{F}'}{\partial t} + \mathscr{G} &= K(t) + \epsilon(\mathscr{F}'^2 - 2\mathscr{F}\mathscr{F}'' - \mathscr{G}^2), \\ E\mathscr{G}'' - p\frac{\partial \mathscr{G}}{\partial t} - \mathscr{F}' &= 2\epsilon(\mathscr{G}\mathscr{F}' - \mathscr{F}\mathscr{G}'),\end{aligned}\right\} \tag{4.4}$$

where a dash now represents differentiation with respect to ζ, and E is the Ekman number,

$$E = \frac{\nu}{\Omega d^2} \ll 1. \tag{4.5}$$

The boundary conditions are

$$\left.\begin{aligned} &\mathscr{F} = \mathscr{F}' = 0, \quad \mathscr{G} = e^{it}, \quad \text{on} \quad \zeta = 0, \\ &\mathscr{F} = \mathscr{F}' = \mathscr{G} = 0, \qquad\quad \text{on} \quad \zeta = \sqrt{2}.\end{aligned}\right\} \tag{4.6}$$

Again, we look for asymptotic expansions for small ϵ by assuming expansions of the form,

$$\mathscr{F} \sim \mathscr{F}_{01}(\zeta)\,e^{it} + \epsilon[\mathscr{F}_{10}(\zeta) + \mathscr{F}_{12}(\zeta)\,e^{2it}] + \dots, \tag{4.7}$$

and similarly for $\mathscr{G}$ and K.

The first approximation is found by the usual means, and the equations are

$$\left.\begin{aligned} E\mathscr{F}_{01}''' - ip\mathscr{F}_{01}' + \mathscr{G}_{01} &= K_{01}, \\ E\mathscr{G}_{01}'' - ip\mathscr{G}_{01} - \mathscr{F}_{01}' &= 0.\end{aligned}\right\} \tag{4.8}$$

The boundary conditions are identical with the boundary conditions (4.6). The solution is

$$\left.\begin{aligned}\mathscr{F}_{01} &= -\frac{ipK_{01}}{1-p^2}+A\cosh\frac{\alpha z}{\sqrt{E}}+B\cosh\frac{\alpha(\sqrt{2}-z)}{\sqrt{E}}+C\cosh\frac{\beta z}{\sqrt{E}}\\ &\quad +D\cosh\frac{\beta(\sqrt{2}-z)}{\sqrt{E}}+M,\\ \mathscr{G}_{01} &= \frac{1}{1-p^2}K_{01}-A\frac{i\alpha}{\sqrt{E}}\sinh\frac{\alpha z}{\sqrt{E}}+B\frac{i\alpha}{\sqrt{E}}\sinh\frac{\alpha(\sqrt{2}-z)}{\sqrt{E}}\\ &\quad +C\frac{i\beta}{\sqrt{E}}\sinh\frac{\beta z}{\sqrt{E}}-D\frac{i\beta}{\sqrt{E}}\sinh\frac{\beta(\sqrt{2}-z)}{\sqrt{E}},\end{aligned}\right\}\tag{4.9a}$$

where α and β were defined in §3, and

$$\begin{aligned}K_{01}&\left[\frac{p\sqrt{2}}{1-p^2}-\frac{\sqrt{E}\{1-\cosh\alpha\sqrt{(2/E)}\}}{(1+p)\alpha\sinh\alpha\sqrt{(2/E)}}+\frac{\sqrt{E}\{1-\cos\beta\sqrt{(2/E)}\}}{(1-p)\beta\sinh\beta\sqrt{(2/E)}}\right]\\ &=\left[\frac{\sqrt{E}\{1-\cosh\beta\sqrt{(2/E)}\}}{2\beta\sinh\beta\sqrt{(2/E)}}-\frac{\sqrt{E}\{1-\cos\alpha\sqrt{(2/E)}\}}{2\alpha\sinh\alpha\sqrt{(2/E)}}\right],\end{aligned}\tag{4.9b}$$

and

$$\left.\begin{aligned}A &= \frac{i\sqrt{E}K_{01}}{2(1+p)\alpha\sinh\alpha\sqrt{(2/E)}},\quad B=\frac{i\sqrt{E}}{2\alpha\sinh\alpha\sqrt{(2/E)}}\left[-1+\frac{K_{01}}{1+p}\right],\\ C &= \frac{i\sqrt{E}K_{01}}{2(1-p)\beta\sinh\beta\sqrt{(2/E)}},\quad D=\frac{i\sqrt{E}}{2\beta\sinh\beta\sqrt{(2/E)}}\left[1-\frac{K_{01}}{1-p}\right],\\ M &= A+B\cosh\alpha\sqrt{(2/E)}-C-D\cosh\beta\sqrt{(2/E)}.\end{aligned}\right\}\tag{4.9c}$$

We shall first consider this solution in the limit of $E\to 0$ subject to.

$$p^2 \gg E. \tag{4.10}$$

The exponential terms of the solution represent shear layers near to the disks. Near the lower disk perturbation velocities of $O(1)$ are produced radially and azimuthally corresponding to the $O(1)$ perturbation on the motion of the disk. To satisfy continuity this produces an axial velocity of $O(\sqrt{E})$. Just beyond the lower boundary layer, the respective velocities are:

$$\left.\begin{aligned}\text{azimuthal} &\sim \frac{1}{2p}\sqrt{\frac{E}{2}}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)+O(E),\\ \text{radial} &\sim \frac{i}{2}\sqrt{\frac{E}{2}}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)+O(E),\\ \text{axial} &\sim \frac{i\sqrt{E}}{2}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)+O(E).\end{aligned}\right\}\tag{4.11}$$

The azimuthal and radial velocities persist to the far boundary. Since there is no perturbation on the motion of the far disk, the azimuthal and radial velocities will also be $O(\sqrt{E})$ in the upper shear layer. This means that the axial velocity in the upper shear layer will be $O(E)$. The change in magnitude from the $O(\sqrt{E})$ velocity at the edge of the lower boundary layer is brought about by the pressure gradient, which is given by

$$K_{01} \sim \frac{1-p^2}{2p}\sqrt{\frac{E}{2}}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)+O(E). \tag{4.12}$$

It is not difficult to verify that this limiting solution, taken subject to (4.10) corresponds to the single-disk case previously considered. The shear layer at the lower disk is properly regarded as a modified Stokes layer. The solution should be contrasted with the one obtained in the limit $E \to 0$, when

$$p^2 \ll E. \tag{4.13}$$

The structure of the solution is then quite different. In the lower shear layer, the radial and azimuthal velocities are again $O(1)$, and the axial velocity again $O(\sqrt{E})$. However, just beyond the shear layer, the radial velocity falls to

$$\text{radial velocity} \sim \frac{ip}{2} \ll O(\sqrt{E}), \tag{4.14a}$$

while the azimuthal velocity remains of $O(1)$,

$$\text{azimuthal velocity} \sim \tfrac{1}{2}. \tag{4.14b}$$

The axial velocity is given by

$$\text{axial velocity} \sim \frac{i\sqrt{E}}{4}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right). \tag{4.14c}$$

All of these velocities persist as far as the outer edge of the upper shear layer. In the upper shear layer, the velocities are brought to rest, and this necessitates velocities there of corresponding order to the velocities in the lower shear layer. The difference between this case and the last one is the strong action of the Coriolis force, which prevents fluid from being thrown out radially in the interior. Radial velocities of $O(1)$ can exist only in the shear layers, where the viscous force is important. The action of the accelerative force is always small. Thus, in this case, the shear layers are properly thought of as modified Ekman layers. In the limit $E \to 0$, it does not correspond to the single-disk case previously considered. The correct boundary condition for the single-disk problem to make it correspond is

$$\mathscr{G} \to \tfrac{1}{2} \quad \text{as} \quad \zeta \to \infty, \tag{4.15}$$

and not $\mathscr{G} \to 0$ as $\zeta \to \infty$. It is not then possible to eliminate the pressure gradient from the equations of motion by using (2.9).

5. The second-order steady solution

We can conclude from the above discussion that when p is not actually equal to zero, the two-disks problem taken in the actual limit of $E = 0$ will correspond to the single-disk problem to first order in ϵ. The question remains to be asked whether the two problems correspond completely when $p \neq 0$ or whether they do not. The answer is, that they correspond only as far as the first asymptotic approximation and no further. If we examine the second-order equations, it can be seen that the same type of discontinuity will always exist for the steady velocity components. Their equations are

$$\left.\begin{aligned} E\mathscr{F}'''_{10} + \mathscr{G}_{10} &= K_{10} + \text{steady part}\,\{\mathscr{F}'^2_{01} - 2\mathscr{F}_{01}\mathscr{F}''_{01} - \mathscr{G}^2_{01}\}, \\ E\mathscr{G}''_{10} - \mathscr{F}'_{10} &= 2\ \text{steady part}\,\{\mathscr{G}_{01}\mathscr{F}'_{01} - \mathscr{F}_{01}\mathscr{G}'_{01}\}. \end{aligned}\right\} \tag{5.1}$$

The homogeneous part of these equations corresponds exactly to the first-order equations (4.8) when $p = 0$. Thus, away from the shear layers, the Coriolis force will dominate the motion, and prevent any radial motion. The result will be a finite, $O(\epsilon)$, azimuthal velocity and radial pressure gradient in the interior.

For simplicity, suppose we consider the problem in the limit of $E \to 0$. This simplifies the algebra and the solution takes the form

$$\left.\begin{aligned} \mathscr{F}_{01} &\sim \frac{\sqrt{E}}{4}\left[\mathscr{P}\left(\frac{z}{\sqrt{E}};p\right)+a\,e^{(\mu/\sqrt{E})z}+b\,e^{(\lambda/\sqrt{E})z}+c\,e^{-(\mu/\sqrt{E})z}\right. \\ &\qquad \left.+d\,e^{-(\lambda/\sqrt{E})z}+m\right]+O(E), \\ \mathscr{G}_{01} &\sim \frac{1}{4}\left[\mathscr{Q}\left(\frac{z}{\sqrt{E}};p\right)-i\mu a\,e^{(\mu/\sqrt{E})z}+i\lambda b\,e^{(\lambda/\sqrt{E})z}\right. \\ &\qquad \left.-i\mu c\,e^{-(\mu/\sqrt{E})z}-i\lambda d e^{-(\lambda/\sqrt{E})z}+4K_{10}\right]+O(\sqrt{E}), \end{aligned}\right\} \tag{5.2a}$$

where $\mathscr{P}$ and $\mathscr{Q}$ are particular integrals of (5.1) and correspond to the particular integrals obtained in the single-disk solution. μ and λ are defined by

$$\mu = -\frac{1+i}{\sqrt{2}}, \quad \lambda = -\frac{1-i}{\sqrt{2}}, \tag{5.2b}$$

and a, b, c, d and m are constants for any value of p. They and K_{10} have to be determined by the six homogeneous boundary conditions, namely,

$$\mathscr{F}_{01} = \mathscr{F}'_{01} = \mathscr{G}_{01} = 0; \quad \zeta = 0, \quad \zeta = \sqrt{2}. \tag{5.3}$$

The two important results that are derived by this are

$$\left.\begin{aligned} m &= (1/2\sqrt{2})(\sqrt{2}n_1+n_2+n_3), \\ K_{10} &= \tfrac{1}{8}(\sqrt{2}n_1+n_2+n_3), \end{aligned}\right\} \tag{5.4a}$$

where

$$\left.\begin{aligned} n_1(p) &= -\mathscr{P}(0;p), \\ n_2(p) &= -\mathscr{P}'(0;p), \\ n_3(p) &= -\mathscr{Q}(0;p). \end{aligned}\right\} \tag{5.4b}$$

It follows from this that a necessary condition for K_{10} (and hence the interior azimuthal velocity) to reduce to zero is that there is no axial flow in the interior ($m = 0$). This corresponds to having $C(p)$ of (3.7) equal to zero, which only occurs for one value of p, $p \approx 4{\cdot}3$. Otherwise the results of the one-disk problem and the two-disks problem will not agree. They can be brought into agreement, however, by rotating the upper disk with a steady (dimensionless) perturbation velocity of $\epsilon\Gamma$. It is possible to choose $\Gamma = \Gamma_0(p)$ so that K_{10} is zero. The condition for this is

$$\Gamma_0(p) = -\tfrac{1}{4}(\sqrt{2}n_1+n_2+n_3), \tag{5.5}$$

when the steady outflow from the boundary layer is

$$\frac{\epsilon\sqrt{E}}{4\sqrt{2}}(\sqrt{2}n_1+n_2+n_3), \tag{5.6}$$

which, except for a dimensional scaling, equals $C(p)$.

Of course, this gives agreement only to $O(\epsilon)$, and the singularity will reappear in the steady $O(\epsilon^3)$ terms, and indeed recur throughout the asymptotic expansion. Presumably by defining the perturbation velocity Γ_0 as a suitable power series in ϵ,

$$\Gamma_0 = \Gamma_{00} + \epsilon^2\Gamma_{02} + \epsilon^4\Gamma_{04} + \ldots,$$

the azimuthal velocity could always be made to approach zero in the interior, and so achieve complete correspondence between the single-disk and two-disk problems. This is rather an artificial case, however, and it can be concluded that, when studying the single-disk case, one should allow a steady, azimuthal velocity with an arbitrarily prescribed value in the far field. The actual value will depend on the rotational speed of the far boundary, and a zero perturbation velocity there will not necessarily imply a zero interior velocity (although it will imply that the interior velocity is $O(\epsilon)$ for most values of p). In a subsequent paper (Jones 1969) it will be shown that this steady far-field velocity, although small, can be very important when one considers the resonance point $p = 1$.

This work was done at Imperial College, London, and was part of a Ph.D. thesis submitted to the University of London. I should like to express my deepest gratitude to Dr S. Rosenblat, who communicated to me his enthusiasm for a very interesting subject. I should also like to thank Dr D. L. Misell for computing assistance, and to acknowledge the receipt of a grant from the Science Research Council during the period in which the work was executed.

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